BOARD NOTES

18 OCTOBER 2018

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CHAPTER 3 –
POLYNOMIAL AND RATIONAL FUNCTIONS

- SECTION 3.4 - ZEROS OF POLYNOMIAL FUNCTIONS

Objectives:

- Use the remainder and factor theorem
- Use Descartes' rule of signs to determine the number of positive and negative real zeros of a polynomial function
- Find the real zeros of a polynomial function
- Solve polynomial equations
- Use the Intermediate Value Theorem

Division Algorithm

If $f(x)$ and $g(x)$ denote polynomial functions and if $g(x)$ is a polynomial whose degree is greater than zero, then there are unique polynomial functions $q(x)$ and $r(x)$ such that

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)} \quad \text{or} \quad \underset{\text{dividend}}{f(x)} = \underset{\text{quotient}}{q(x)}\,\underset{\text{divisor}}{g(x)} + \underset{\text{remainder}}{r(x)} \qquad \textbf{(1)}$$

where $r(x)$ is either the zero polynomial or a polynomial of degree less than that of $g(x)$.

Remainder Theorem

Let f be a polynomial function. If $f(x)$ is divided by $x - c$, then the remainder is $f(c)$.

Factor Theorem

Let f be a polynomial function. Then $x - c$ is a factor of $f(x)$ if and only if $f(c) = 0$.

1. If $f(c) = 0$, then $x - c$ is a factor of $f(x)$.
2. If $x - c$ is a factor of $f(x)$, then $f(c) = 0$.

Number of Real Zeros

A polynomial function cannot have more real zeros than its degree.

Descartes' Rule of Signs

Let f denote a polynomial function written in standard form.

The number of positive real zeros of f either equals the number of variations in the sign of the nonzero coefficients of $f(x)$ or else equals that number less an even integer.

The number of negative real zeros of f either equals the number of variations in the sign of the nonzero coefficients of $f(-x)$ or else equals that number less an even integer.

Rational Zeros Theorem

Let f be a polynomial function of degree 1 or higher of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \qquad a_n \neq 0 \quad a_0 \neq 0$$

where each coefficient is an integer. If $\frac{p}{q}$, in lowest terms, is a rational zero of f, then p must be a factor of a_0, and q must be a factor of a_n.

$$Possible\ Rational\ Zeros\ (PRZ) = \frac{p\ which\ is\ all\ factors\ of\ a_0}{q\ which\ is\ all\ factors\ of\ a_n}$$

Steps for Finding the Real Zeros of a Polynomial Function

STEP 1: Use the degree of the polynomial to determine the maximum number of real zeros.

STEP 2: Use Descartes' Rule of Signs to determine the possible number of positive zeros and negative zeros.

STEP 3: (a) If the polynomial has integer coefficients, use the Rational Zeros Theorem to identify those rational numbers that potentially could be zeros.

(b) Use substitution, synthetic division, or long division to test each potential rational zero. Each time that a zero (and thus a factor) is found, repeat Step 3 on the depressed equation.

In attempting to find the zeros, remember to use (if possible) the factoring techniques that you already know (special products, factoring by grouping, and so on).

Intermediate Value Theorem

Let f denote a polynomial function. If $a < b$ and if $f(a)$ and $f(b)$ are of opposite sign, there is at least one real zero of f between a and b.

Approximating the Real Zeros of a Polynomial Function

STEP 1: Find two consecutive integers a and $a + 1$ such that f has a zero between them.

STEP 2: Divide the interval $[a, a + 1]$ into 10 equal subintervals.

STEP 3: Evaluate f at each endpoint of the subintervals until the Intermediate Value Theorem applies; this interval then contains a zero.

STEP 4: Now divide the new interval into 10 equal subintervals and repeat Step 3.

STEP 5: Continue with Steps 3 and 4 until the desired accuracy is achieved.

Complex Polynomial

A variable in the complex number system is referred to as a **complex variable.** A **complex polynomial function** f of degree n is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \qquad (1)$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are complex numbers, $a_n \neq 0$, n is a nonnegative integer, and x is a complex variable. As before, a_n is called the **leading coefficient** of f. A complex number r is called a **complex zero** of f if $f(r) = 0$.

Fundamental Theorem of Algebra (FTA)

Every complex polynomial function f of degree $n \geq 1$ has at least one complex zero.

Every complex polynomial function f of degree $n \geq 1$ can be factored into n linear factors (not necessarily distinct) of the form

$$f(x) = a_n(x - r_1)(x - r_2) \cdot \cdots \cdot (x - r_n) \qquad (2)$$

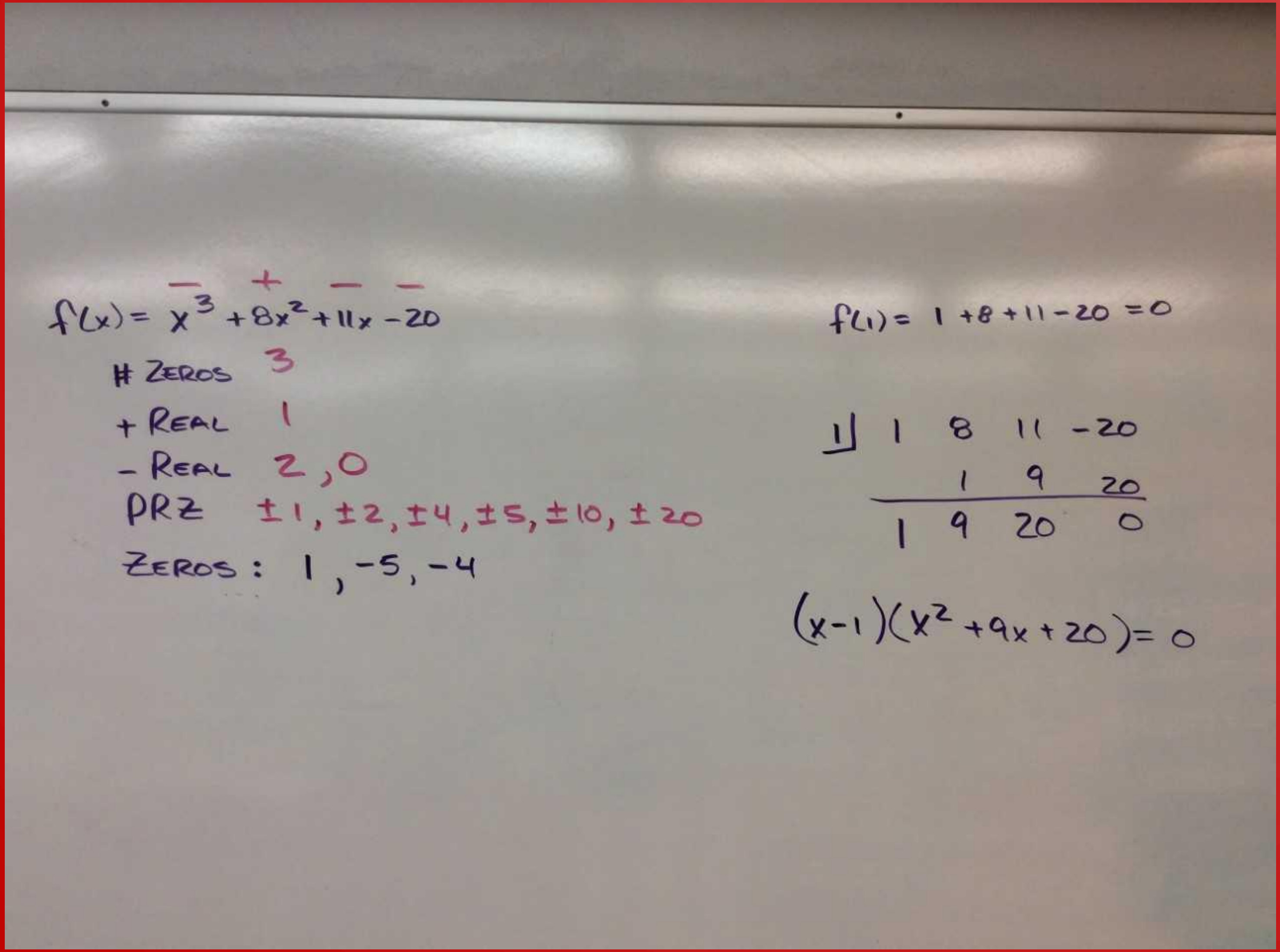
where $a_n, r_1, r_2, \ldots, r_n$ are complex numbers. That is, every complex polynomial function of degree $n \geq 1$ has exactly n complex zeros, some of which may repeat.

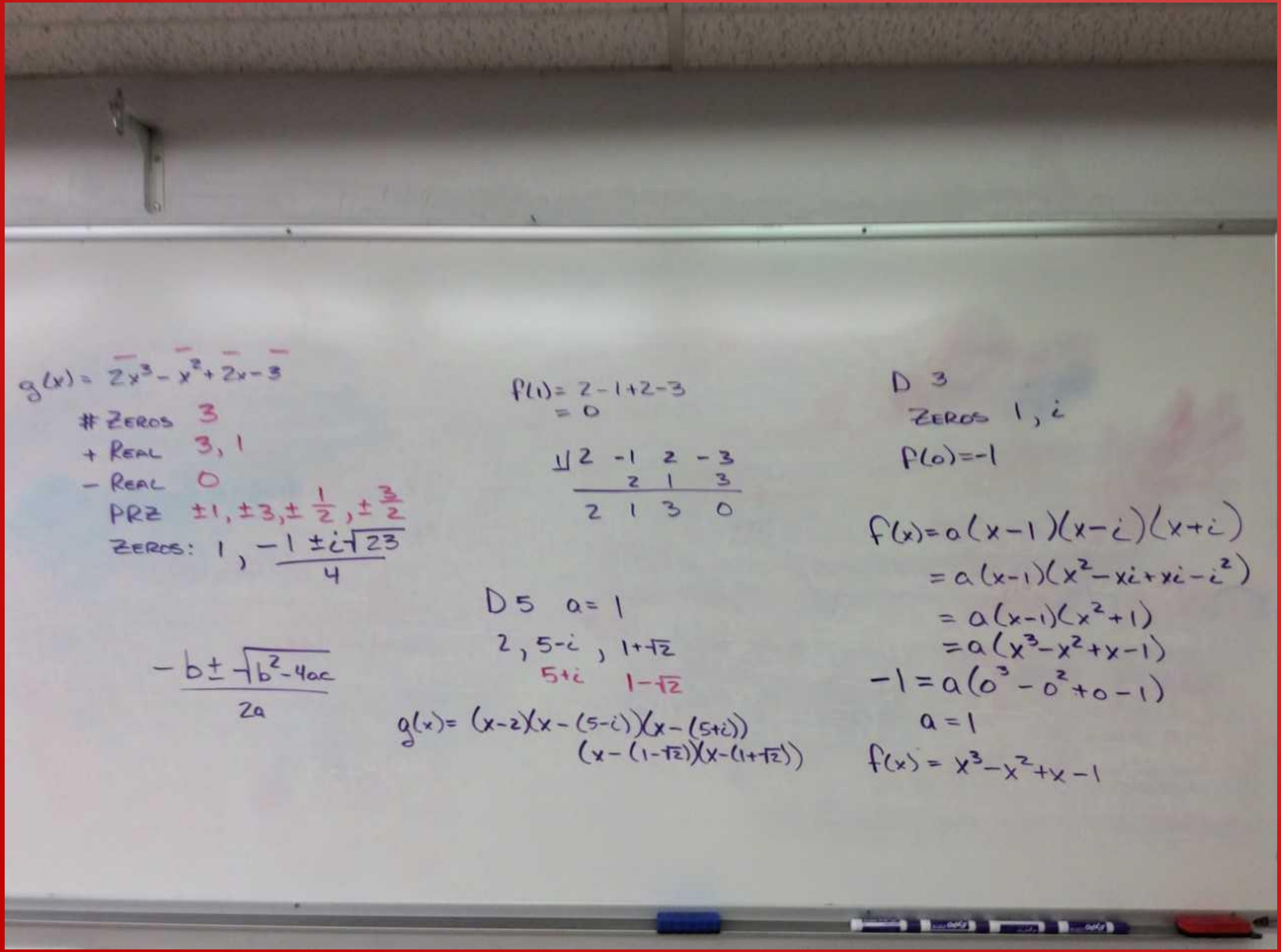
Conjugate Pairs Theorem

Let f be a polynomial function whose coefficients are real numbers. If $r = a + bi$ is a zero of f, the complex conjugate $\bar{r} = a - bi$ is also a zero of f.

Corollary

A polynomial function f of odd degree with real coefficients has at least one real zero.


− + − −
f(x) = x^3 + 8x^2 + 11x − 20
ZEROS 3
+ REAL 1
− REAL 2, 0
PRZ ±1, ±2, ±4, ±5, ±10, ±20
ZEROS : 1, −5, −4
f(1) = 1 + 8 + 11 − 20 = 0
1 | 1 8 11 −20
1 9 20
1 9 20 0
(x−1)(x^2 + 9x + 20) = 0


g(x) = 2x³ − x² + 2x − 3
Zeros 3
+ Real 3, 1
− Real 0
PRZ ±1, ±3, ±1/2, ±3/2
Zeros: 1, (−1 ± i√23)/4
(−b ± √(b² − 4ac))/2a
f(1) = 2 − 1 + 2 − 3
= 0
1| 2 −1 2 −3
2 1 3
2 1 3 0
D 5 a = 1
2, 5−i, 1+√2
5+i 1−√2
g(x) = (x−2)(x − (5−i))(x − (5+i))
(x − (1−√2))(x − (1+√2))
D 3
Zeros 1, i
f(0) = −1
f(x) = a(x−1)(x−i)(x+i)
= a(x−1)(x² − xi + xi − i²)
= a(x−1)(x²+1)
= a(x³ − x² + x − 1)
−1 = a(0³ − 0² + 0 − 1)
a = 1
f(x) = x³ − x² + x − 1

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CHAPTER 3 –
POLYNOMIAL AND RATIONAL FUNCTIONS

- SECTION 3.5 - RATIONAL FUNCTIONS AND THEIR GRAPHS

Objectives:

- Find the domain of a rational function
- Find the vertical asymptotes of a rational function
- Find the horizontal or oblique (slant) asymptote of a rational function
- Graph rational functions
- Solve applied problems

A **rational function** is a function of the form

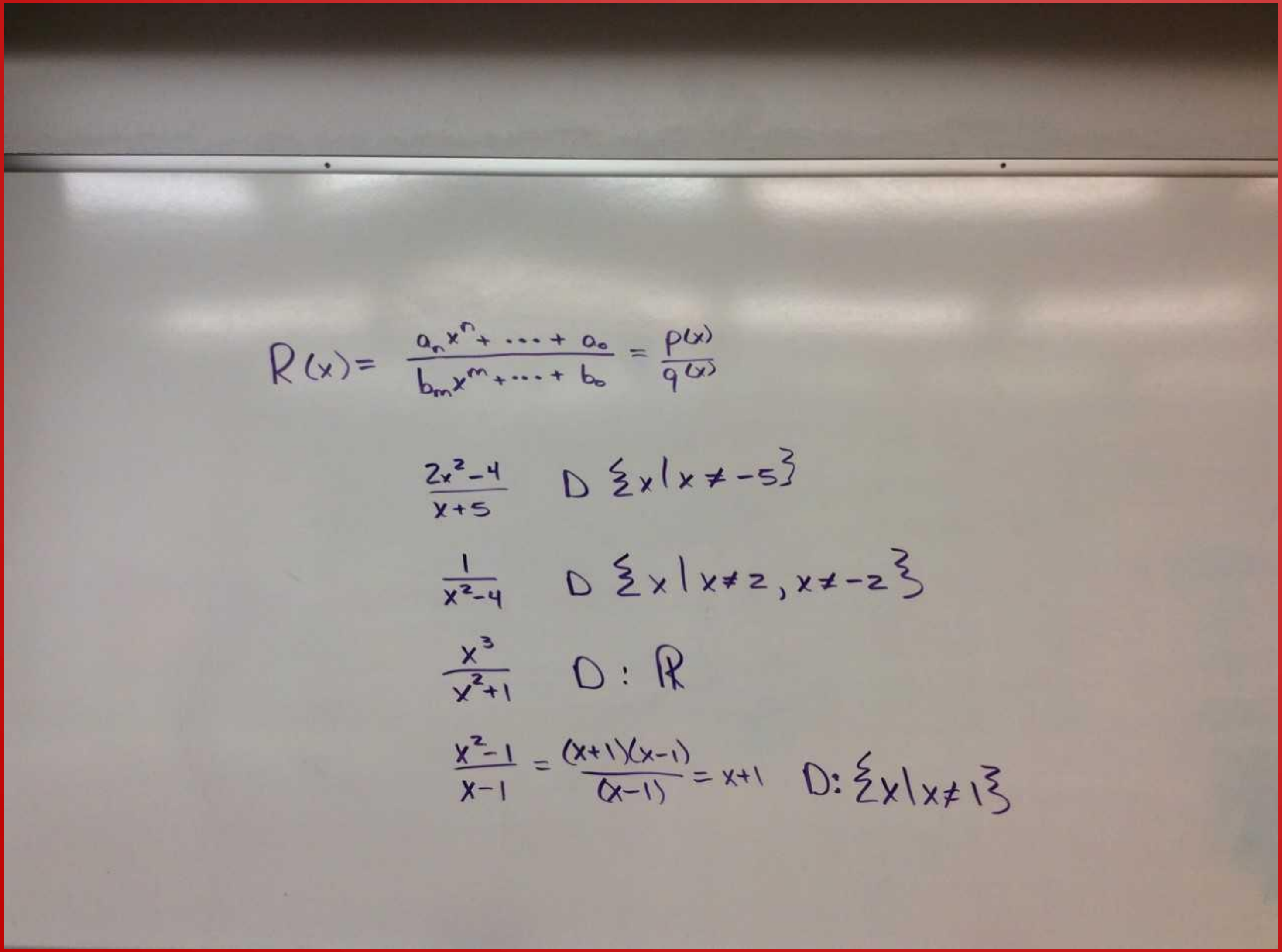
$$R(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomial functions and q is not the zero polynomial. The domain of a rational function is the set of all real numbers except those for which the denominator q is 0.

Let R denote a function.

If, as $x \to -\infty$ or as $x \to \infty$, the values of $R(x)$ approach some fixed number L, then the line $y = L$ is a **horizontal asymptote** of the graph of R. [Refer to Figures 27(a) and (b) on page 192.]

If, as x approaches some number c, the values $|R(x)| \to \infty$ [that is, $R(x) \to -\infty$ or $R(x) \to \infty$], then the line $x = c$ is a **vertical asymptote** of the graph of R. [Refer to Figures 27(c) and (d).]



FRANK PHILLIPS COLLEGE Plainsmen

BORGER INDEPENDENT SCHOOL DISTRICT 1926

Locating Vertical Asymptotes

A rational function $R(x) = \frac{p(x)}{q(x)}$, *in lowest terms*, will have a vertical asymptote $x = r$ if r is a real zero of the *denominator* q. That is, if $x - r$ is a factor of the denominator q of a rational function $R(x) = \frac{p(x)}{q(x)}$, in lowest terms, R will have the vertical asymptote $x = r$.

Finding a Horizontal or Oblique Asymptote of a Rational Function

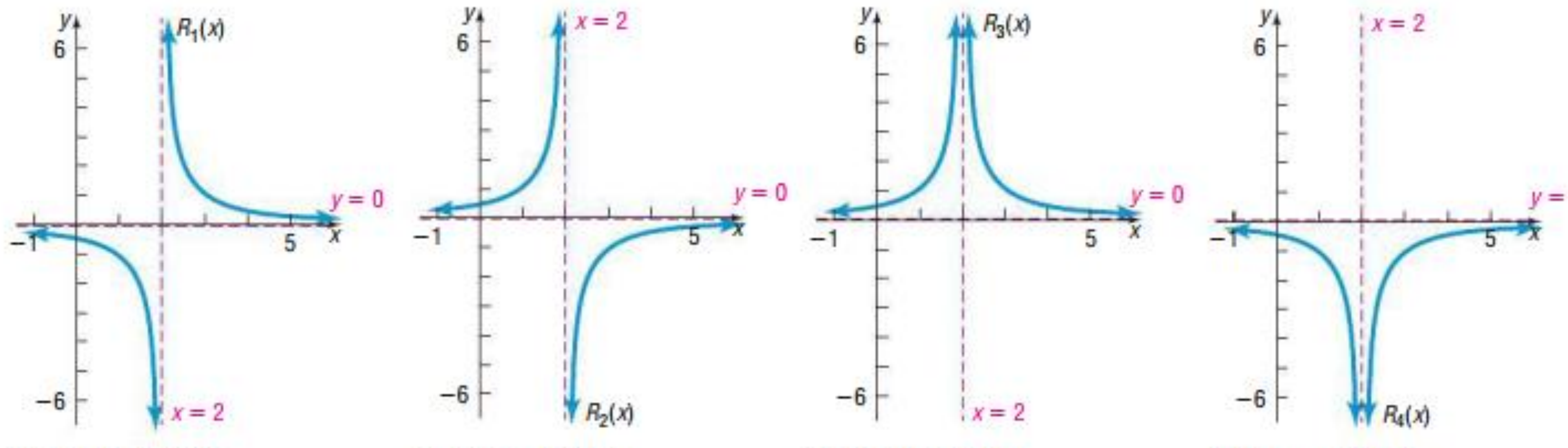
Consider the rational function

$$R(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$$

in which the degree of the numerator is n and the degree of the denominator is m.

1. If $n < m$ (the degree of the numerator is less than the degree of the denominator), the line $y = 0$ is a horizontal asymptote.
2. If $n = m$ (the degree of the numerator equals the degree of the denominator), the line $y = \frac{a_n}{b_m}$ is a horizontal asymptote. (That is, the horizontal asymptote equals the ratio of the leading coefficients.)
3. If $n = m + 1$ (the degree of the numerator is one more than the degree of the denominator), the line $y = ax + b$ is an oblique asymptote, which is the quotient found using long division.
4. If $n \geq m + 2$ (the degree of the numerator is two or more greater than the degree of the denominator), there are no horizontal or oblique asymptotes. The end behavior of the graph will resemble the power function $y = \frac{a_n}{b_m}x^{n-m}$.

Note: A rational function will never have both a horizontal asymptote and an oblique asymptote. A rational function may have neither a horizontal nor an oblique asymptote.



(a) Odd multiplicity
$\lim_{x \to 2^-} R_1(x) = -\infty$
$\lim_{x \to 2^+} R_1(x) = \infty$

(b) Odd multiplicity
$\lim_{x \to 2^-} R_2(x) = \infty$
$\lim_{x \to 2^+} R_2(x) = -\infty$

(c) Even multiplicity
$\lim_{x \to 2^-} R_3(x) = \infty$
$\lim_{x \to 2^+} R_3(x) = \infty$

(d) Even multiplicity
$\lim_{x \to 2^-} R_4(x) = -\infty$
$\lim_{x \to 2^+} R_4(x) = -\infty$

Analyzing the Graph of a Rational Function R

STEP 1: Factor the numerator and denominator of R. Find the domain of the rational function.

STEP 2: Write R in lowest terms.

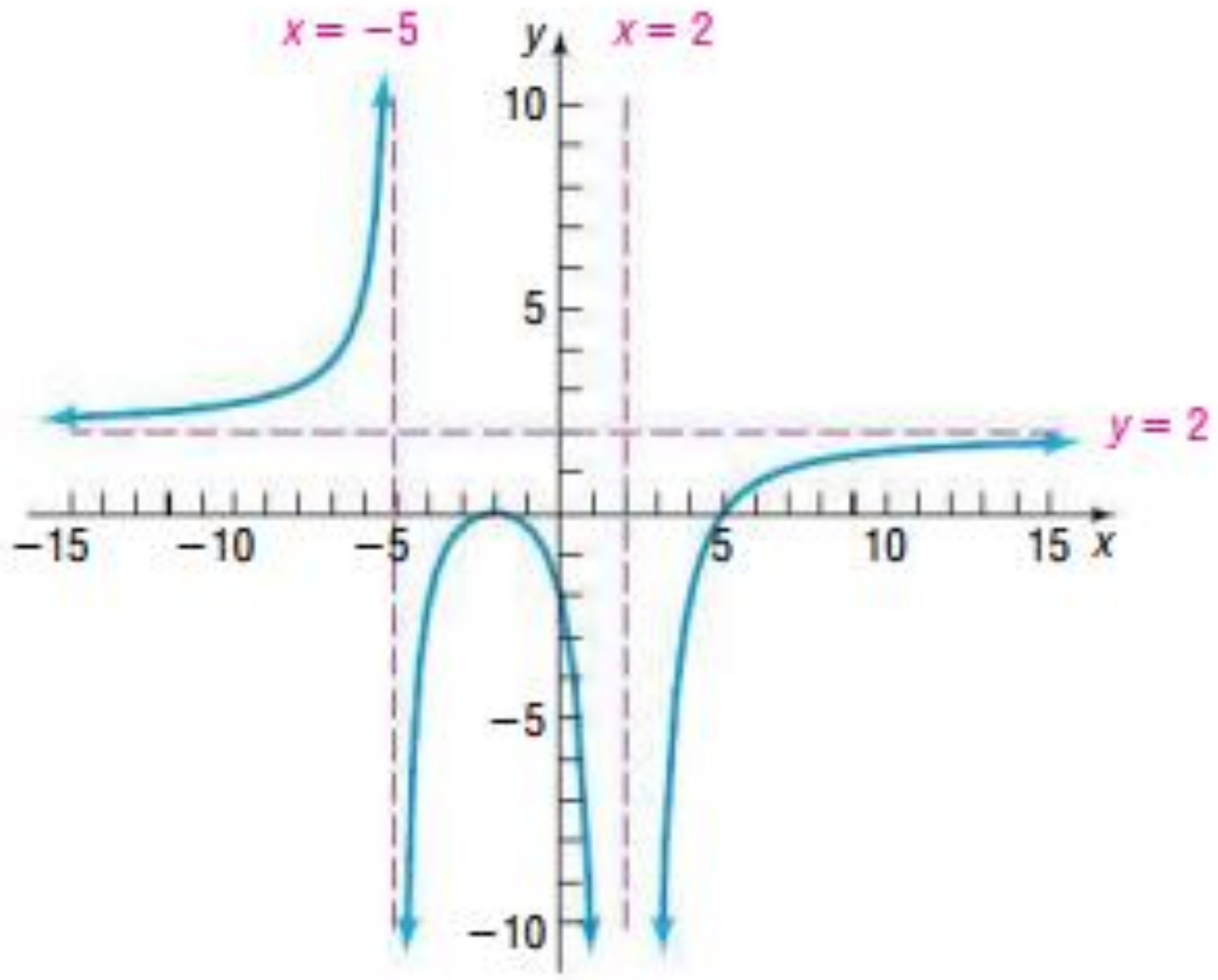
STEP 3: Find and plot the intercepts of the graph. Use multiplicity to determine the behavior of the graph of R at each x-intercept.

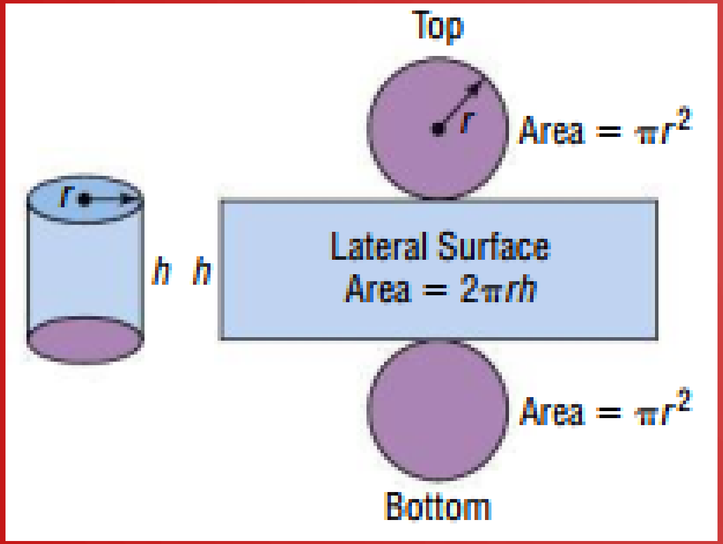
STEP 4: Find the vertical asymptotes. Graph each vertical asymptote using a dashed line. Determine the behavior of the graph of R on either side of each vertical asymptote.

STEP 5: Find the horizontal or oblique asymptote, if one exists. Find points, if any, at which the graph of R intersects this asymptote. Graph the asymptote using a dashed line. Plot any points at which the graph of R intersects the asymptote.

STEP 6: Use the zeros of the numerator and denominator of R to divide the x-axis into intervals. Determine where the graph of R is above or below the x-axis by choosing a number in each interval and evaluating R there. Plot the points found.

STEP 7: Use the results obtained in Steps 1 through 6 to graph R.


x = −5
y
x = 2
10
5
y = 2
−15
−10
−5
5
10
15
x
−5
−10



Finding the Least Cost of a Can

Reynolds Metal Company manufactures aluminum cans in the shape of a cylinder with a capacity of 500 cubic centimeters $\left(\frac{1}{2}\text{ liter}\right)$. The top and bottom of the can are made of a special aluminum alloy that costs 0.05¢ per square centimeter. The sides of the can are made of material that costs 0.02¢ per square centimeter.

(a) Express the cost of material for the can as a function of the radius r of the can.
(b) Use a graphing utility to graph the function $C = C(r)$.
(c) What value of r will result in the least cost?
(d) What is this least cost?