BOARD NOTES

19 OCTOBER 2018

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CHAPTER 3 –
POLYNOMIAL AND RATIONAL FUNCTIONS

- SECTION 3.5 - RATIONAL FUNCTIONS AND THEIR GRAPHS

Objectives:

- Find the domain of a rational function
- Find the vertical asymptotes of a rational function
- Find the horizontal or oblique (slant) asymptote of a rational function
- Graph rational functions
- Solve applied problems

A **rational function** is a function of the form

$$R(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomial functions and q is not the zero polynomial. The domain of a rational function is the set of all real numbers except those for which the denominator q is 0.

Let R denote a function.

If, as $x \to -\infty$ or as $x \to \infty$, the values of $R(x)$ approach some fixed number L, then the line $y = L$ is a **horizontal asymptote** of the graph of R. [Refer to Figures 27(a) and (b) on page 192.]

If, as x approaches some number c, the values $|R(x)| \to \infty$ [that is, $R(x) \to -\infty$ or $R(x) \to \infty$], then the line $x = c$ is a **vertical asymptote** of the graph of R. [Refer to Figures 27(c) and (d).]

Locating Vertical Asymptotes

A rational function $R(x) = \frac{p(x)}{q(x)}$, *in lowest terms*, will have a vertical asymptote $x = r$ if r is a real zero of the *denominator* q. That is, if $x - r$ is a factor of the denominator q of a rational function $R(x) = \frac{p(x)}{q(x)}$, in lowest terms, R will have the vertical asymptote $x = r$.

Finding a Horizontal or Oblique Asymptote of a Rational Function

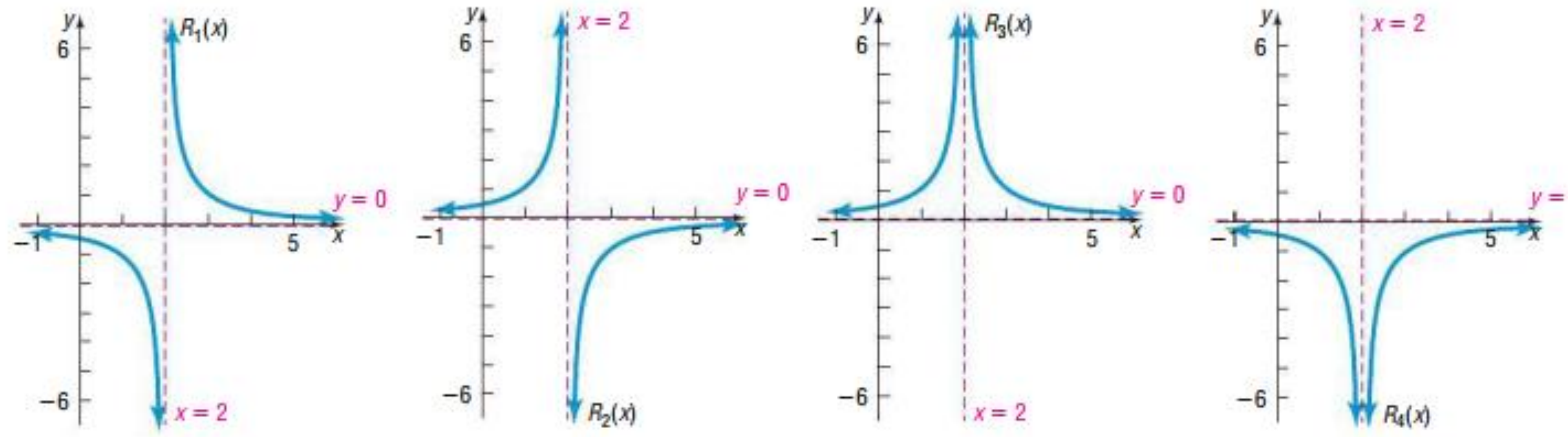
Consider the rational function

$$R(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$$

in which the degree of the numerator is n and the degree of the denominator is m.

1. If $n < m$ (the degree of the numerator is less than the degree of the denominator), the line $y = 0$ is a horizontal asymptote.
2. If $n = m$ (the degree of the numerator equals the degree of the denominator), the line $y = \frac{a_n}{b_m}$ is a horizontal asymptote. (That is, the horizontal asymptote equals the ratio of the leading coefficients.)
3. If $n = m + 1$ (the degree of the numerator is one more than the degree of the denominator), the line $y = ax + b$ is an oblique asymptote, which is the quotient found using long division.
4. If $n \geq m + 2$ (the degree of the numerator is two or more greater than the degree of the denominator), there are no horizontal or oblique asymptotes. The end behavior of the graph will resemble the power function $y = \frac{a_n}{b_m}x^{n-m}$.

Note: A rational function will never have both a horizontal asymptote and an oblique asymptote. A rational function may have neither a horizontal nor an oblique asymptote.



(a) Odd multiplicity
$\lim_{x \to 2^-} R_1(x) = -\infty$
$\lim_{x \to 2^+} R_1(x) = \infty$

(b) Odd multiplicity
$\lim_{x \to 2^-} R_2(x) = \infty$
$\lim_{x \to 2^+} R_2(x) = -\infty$

(c) Even multiplicity
$\lim_{x \to 2^-} R_3(x) = \infty$
$\lim_{x \to 2^+} R_3(x) = \infty$

(d) Even multiplicity
$\lim_{x \to 2^-} R_4(x) = -\infty$
$\lim_{x \to 2^+} R_4(x) = -\infty$

Analyzing the Graph of a Rational Function R

STEP 1: Factor the numerator and denominator of R. Find the domain of the rational function.

STEP 2: Write R in lowest terms.

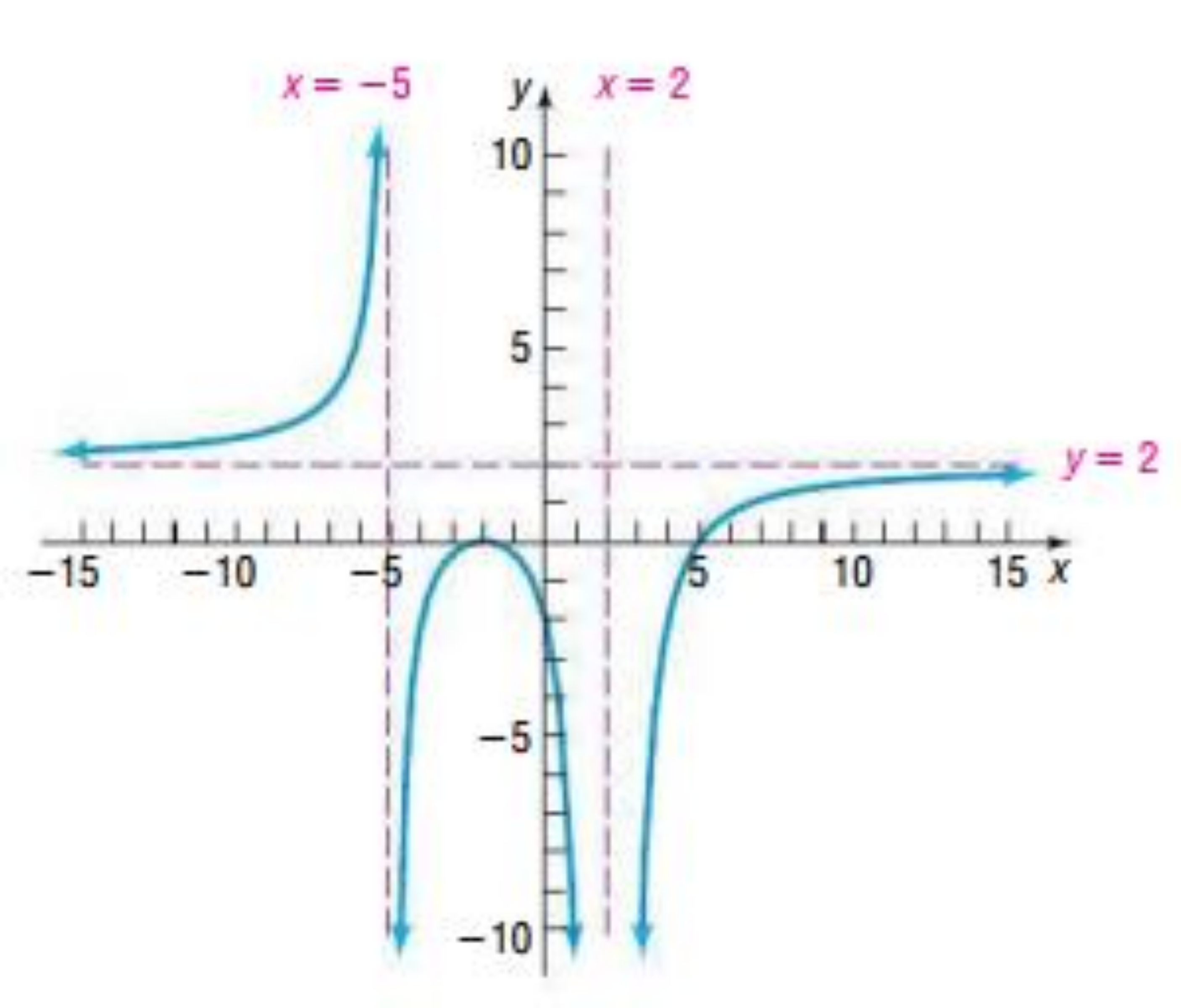
STEP 3: Find and plot the intercepts of the graph. Use multiplicity to determine the behavior of the graph of R at each x-intercept.

STEP 4: Find the vertical asymptotes. Graph each vertical asymptote using a dashed line. Determine the behavior of the graph of R on either side of each vertical asymptote.

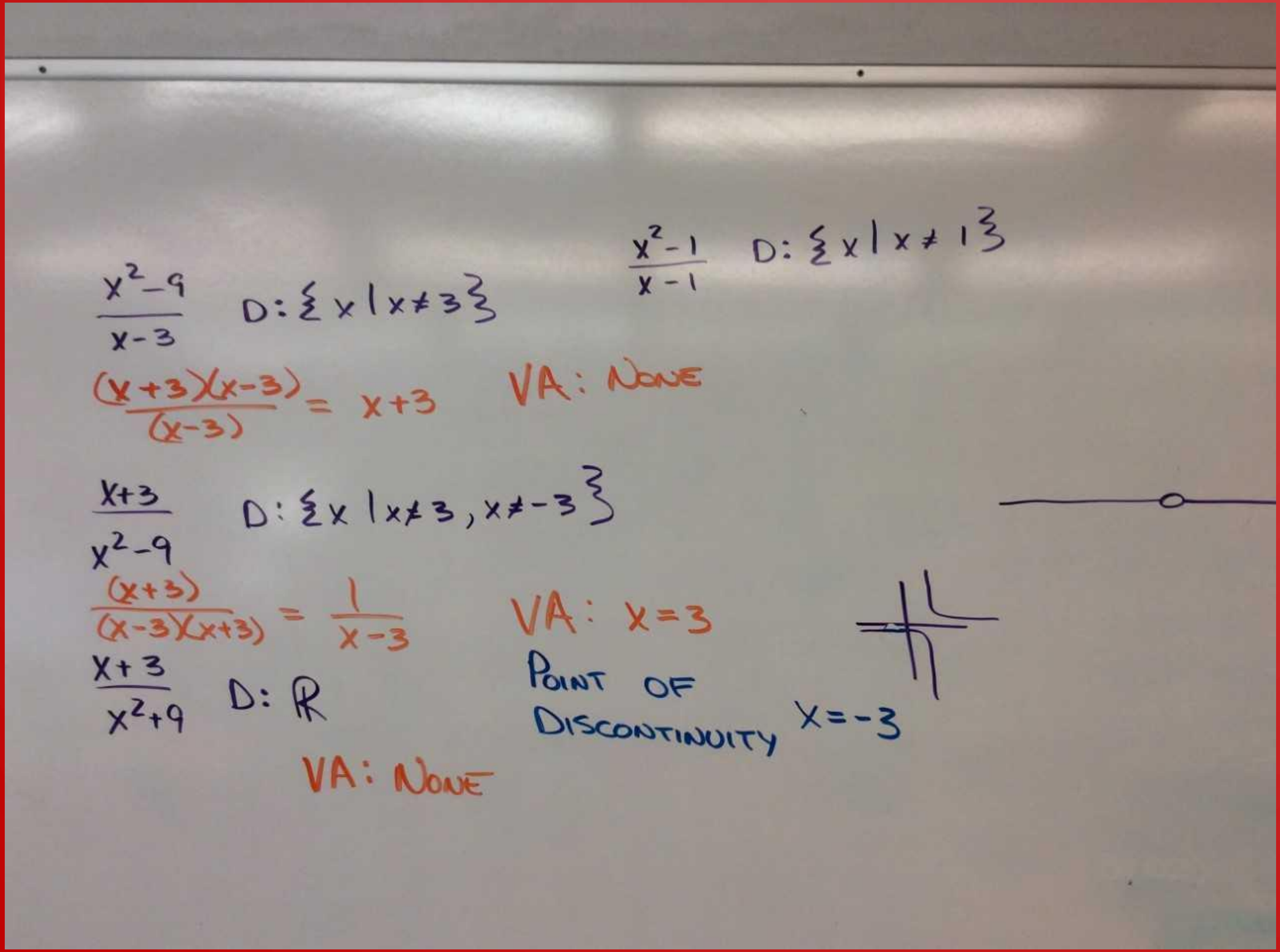
STEP 5: Find the horizontal or oblique asymptote, if one exists. Find points, if any, at which the graph of R intersects this asymptote. Graph the asymptote using a dashed line. Plot any points at which the graph of R intersects the asymptote.

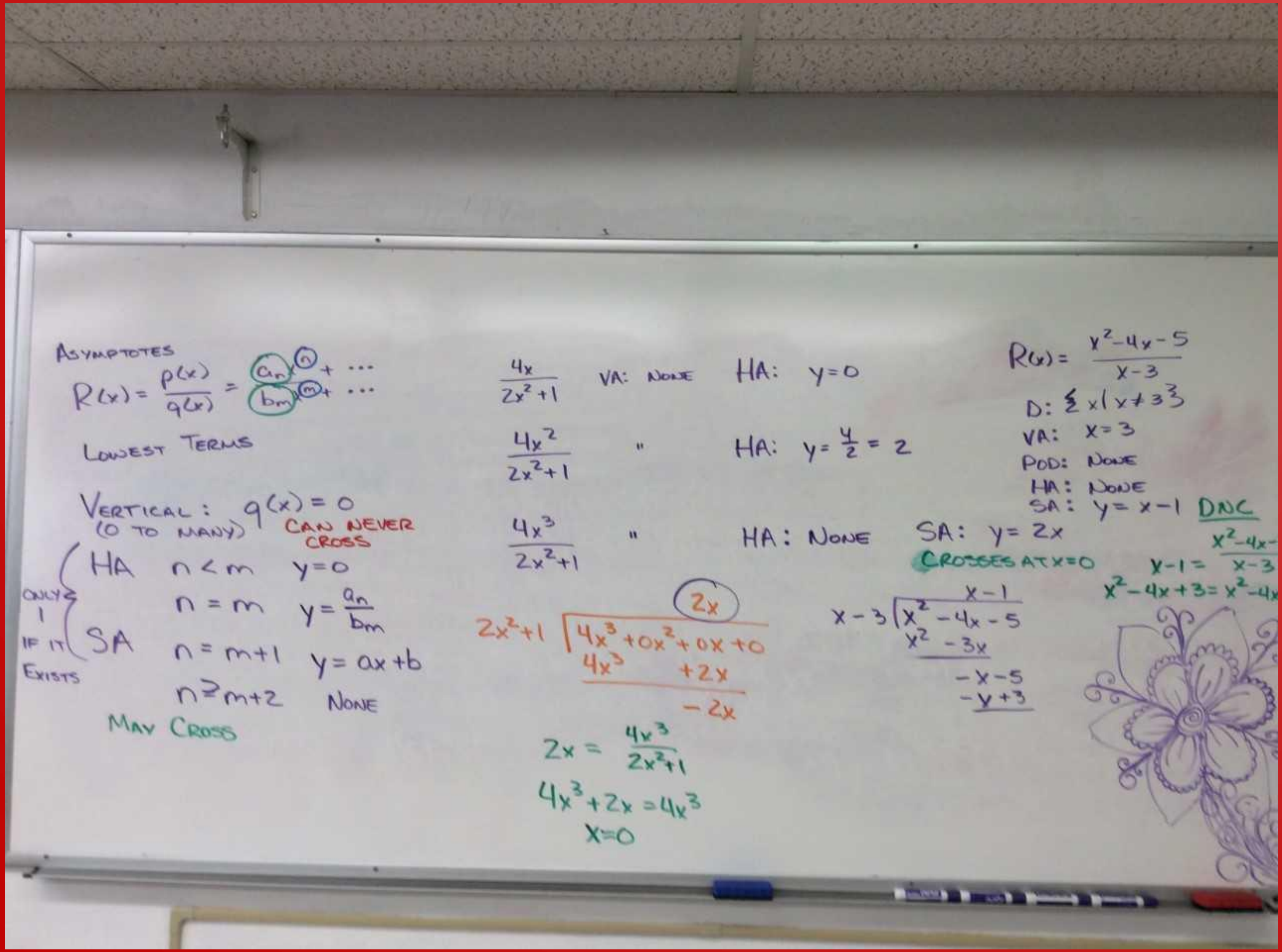
STEP 6: Use the zeros of the numerator and denominator of R to divide the x-axis into intervals. Determine where the graph of R is above or below the x-axis by choosing a number in each interval and evaluating R there. Plot the points found.

STEP 7: Use the results obtained in Steps 1 through 6 to graph R.



x = −5
y
x = 2
10
5
y = 2
−15
−10
−5
5
10
15
x
−5
−10


x²−1 / x−1 D: {x | x ≠ 1}
x²−9 / x−3 D: {x | x ≠ 3}
(x+3)(x−3) / (x−3) = x+3
VA: None
x+3 / x²−9 D: {x | x ≠ 3, x ≠ −3}
(x+3) / (x−3)(x+3) = 1 / x−3
VA: x=3
Point of Discontinuity x=−3
x+3 / x²+9 D: R
VA: None


Asymptotes
R(x) = p(x)/q(x) = aₙ xⁿ + ... / bₘ xᵐ + ...
Lowest Terms
Vertical: q(x) = 0
(0 to many)
Can never cross
HA n < m y = 0
n = m y = aₙ/bₘ
SA n = m+1 y = ax + b
n ≥ m+2 None
Only 1 if it exists
May Cross
4x / 2x²+1 VA: None HA: y = 0
4x² / 2x²+1 " HA: y = 4/2 = 2
4x³ / 2x²+1 " HA: None SA: y = 2x
Crosses at x=0
2x
2x²+1 | 4x³ + 0x² + 0x + 0
4x³ + 2x
− 2x
2x = 4x³ / 2x²+1
4x³ + 2x = 4x³
x=0
R(x) = x² − 4x − 5 / x − 3
D: {x | x ≠ 3}
VA: x = 3
POD: None
HA: None
SA: y = x − 1 DNC
x − 1
x − 3 | x² − 4x − 5
x² − 3x
−x − 5
−x + 3

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CHAPTER 3 –
POLYNOMIAL AND RATIONAL FUNCTIONS

- SECTION 3.6 - POLYNOMIAL AND RATIONAL INEQUALITIES

Objectives:

- Solve polynomial inequalities
- Solve rational inequalities

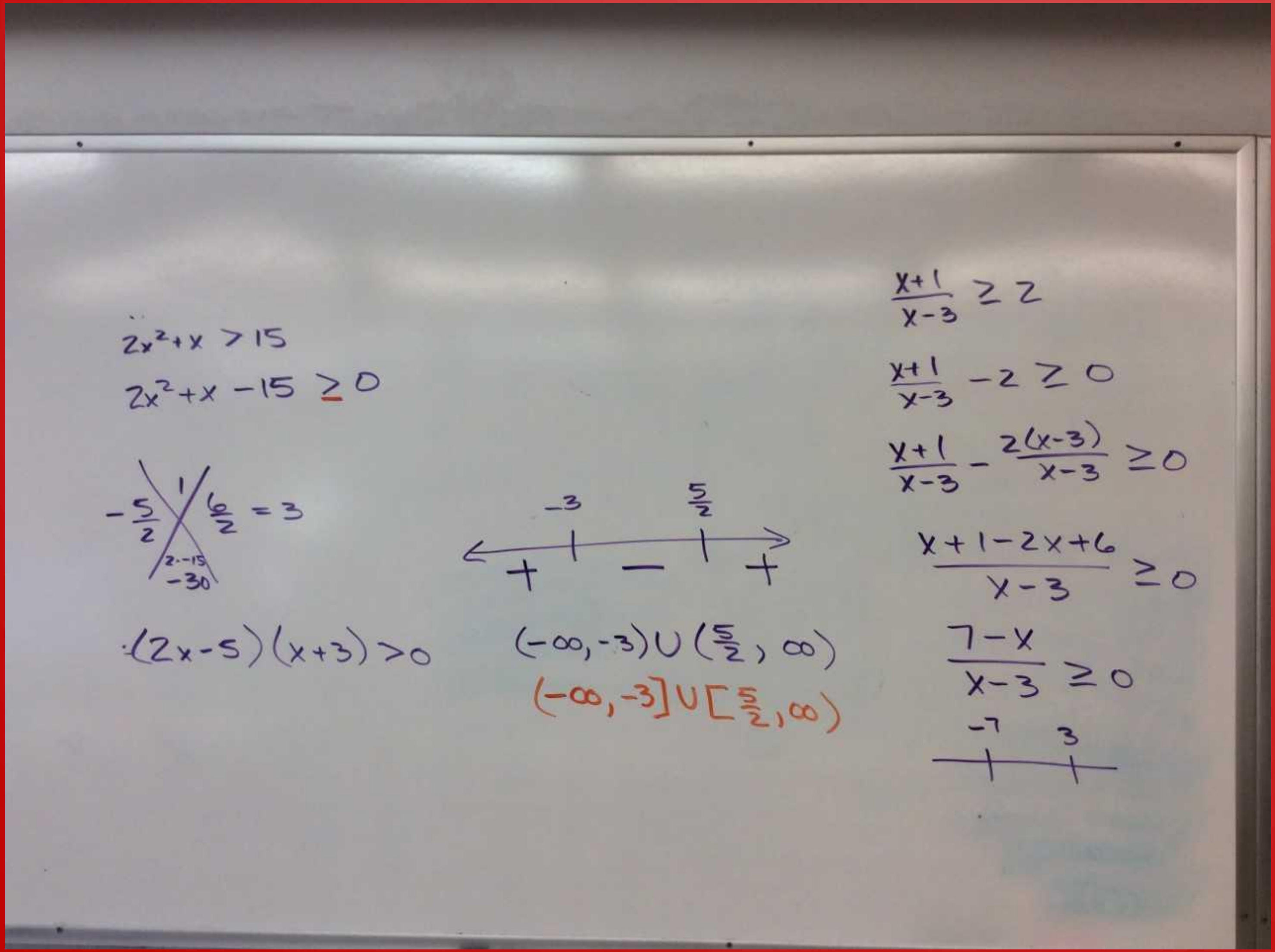
Procedure for Solving Polynomial Inequalities

1. Express the inequality in the form

$$f(x) < 0 \quad \text{or} \quad f(x) > 0,$$

   where f is a polynomial function.
2. Solve the equation $f(x) = 0$. The real solutions are the **boundary points**.
3. Locate these boundary points on a number line, thereby dividing the number line into intervals.
4. Choose one representative number, called a **test value**, within each interval and evaluate f at that number.
   - **a.** If the value of f is positive, then $f(x) > 0$ for all numbers, x, in the interval.
   - **b.** If the value of f is negative, then $f(x) < 0$ for all numbers, x, in the interval.
5. Write the solution set, selecting the interval or intervals that satisfy the given inequality.

This procedure is valid if $<$ is replaced by $\leq$ or $>$ is replaced by $\geq$. However, if the inequality involves $\leq$ or $\geq$, include the boundary points [the solutions of $f(x) = 0$] in the solution set.



2x²+x > 15
2x²+x −15 ≥ 0
−5/2
1
6/2 = 3
2·−15
−30
(2x−5)(x+3)>0
−3
5/2
+ − +
(−∞,−3)∪(5/2,∞)
(−∞,−3]∪[5/2,∞)
(x+1)/(x−3) ≥ 2
(x+1)/(x−3) −2 ≥ 0
(x+1)/(x−3) − 2(x−3)/(x−3) ≥ 0
(x+1−2x+6)/(x−3) ≥ 0
(7−x)/(x−3) ≥ 0
−7
3

The Position Function for a Free-Falling Object Near Earth's Surface

An object that is falling or vertically projected into the air has its height above the ground, $s(t)$, in feet, given by

$$s(t) = -16t^2 + v_0 t + s_0,$$

where v_0 is the original velocity (initial velocity) of the object, in feet per second, t is the time that the object is in motion, in seconds, and s_0 is the original height (initial height) of the object, in feet.

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CHAPTER 3 –
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- SECTION 3.7 - MODELING USING VARIATION

Objectives:

- Solve direct variation problems
- Solve inverse variation problems
- Solve combined variation problems
- Solve problems involving joint variation

Direct Variation

If a situation is described by an equation in the form

$$y = kx,$$

where k is a nonzero constant, we say that **y varies directly as x** or **y is directly proportional to x**. The number k is called the **constant of variation** or the **constant of proportionality**.

Inverse Variation

If a situation is described by an equation in the form

$$y = \frac{k}{x},$$

where k is a nonzero constant, we say that **y varies inversely as x** or **y is inversely proportional to x**. The number k is called the **constant of variation**.

Solving Variation Problems

1. Write an equation that models the given English statement.
2. Substitute the given pair of values into the equation in step 1 and find the value of k, the constant of variation.
3. Substitute the value of k into the equation in step 1.
4. Use the equation from step 3 to answer the problem's question.

The owners of Rollerblades Plus determine that the monthly sales, S, of their skates vary directly as their advertising budget, A, and inversely as the price of the skates, P. When \$60,000 is spent on advertising and the price of the skates is \$40, the monthly sales are 12,000 pairs of rollerblades.

a. Write an equation of variation that describes this situation.

b. Determine monthly sales if the amount of the advertising budget is increased