BOARD NOTES

18 OCTOBER 2018

BORGER INDEPENDENT SCHOOL DISTRICT

1926

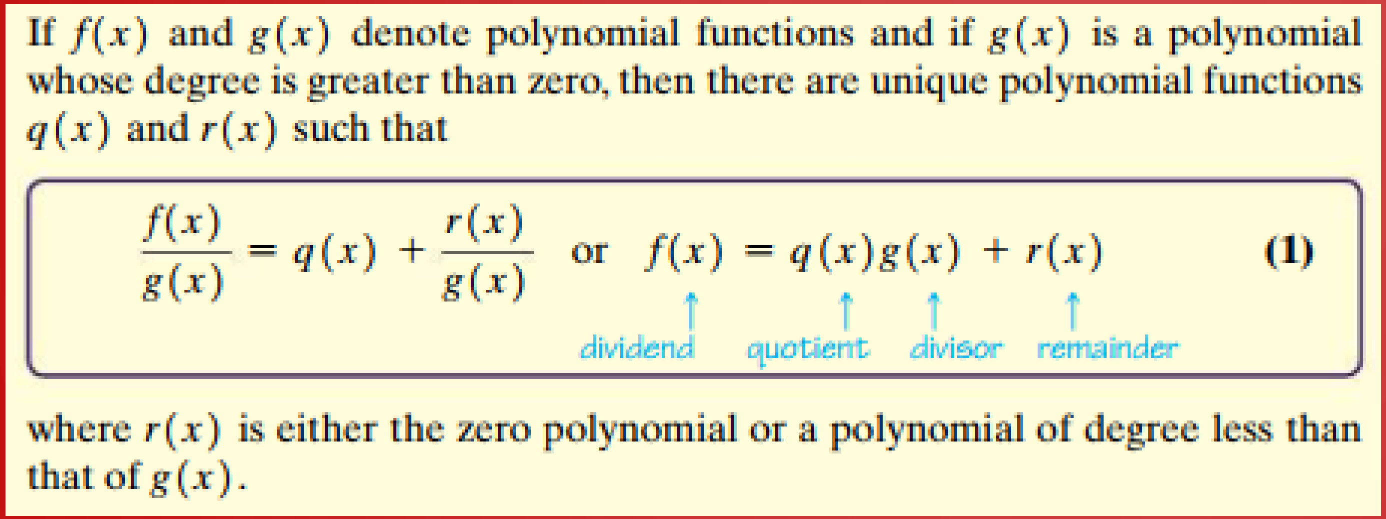
CC PRECALCULUS
CHAPTER 4 –
POLYNOMIAL AND RATIONAL FUNCTIONS

- SECTION 4.6 - COMPLEX ZEROS: FUNDAMENTAL THEOREM OF ALGEBRA

Objectives:

- Use the conjugate pairs theorem
- Find a polynomial with specified zeros
- Find the complex zeros of a polynomial function

Division Algorithm

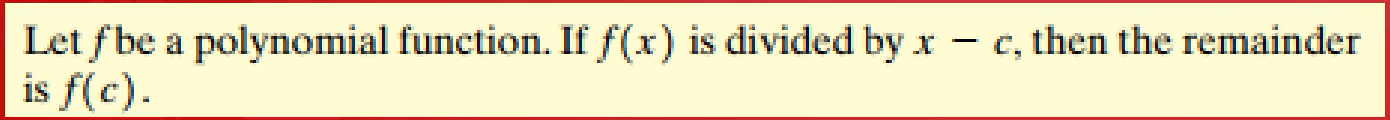


If $f(x)$ and $g(x)$ denote polynomial functions and if $g(x)$ is a polynomial whose degree is greater than zero, then there are unique polynomial functions $q(x)$ and $r(x)$ such that

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)} \quad \text{or} \quad \underset{\underset{\text{dividend}}{\uparrow}}{f(x)} = \underset{\underset{\text{quotient}}{\uparrow}}{q(x)}\underset{\underset{\text{divisor}}{\uparrow}}{g(x)} + \underset{\underset{\text{remainder}}{\uparrow}}{r(x)} \qquad \textbf{(1)}$$

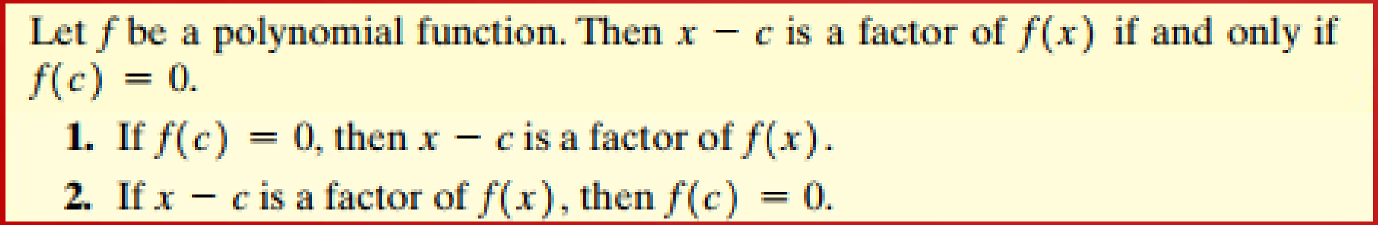
where $r(x)$ is either the zero polynomial or a polynomial of degree less than that of $g(x)$.

Remainder Theorem



Let f be a polynomial function. If $f(x)$ is divided by $x - c$, then the remainder is $f(c)$.

Factor Theorem



Let f be a polynomial function. Then $x - c$ is a factor of $f(x)$ if and only if $f(c) = 0$.

1. If $f(c) = 0$, then $x - c$ is a factor of $f(x)$.
2. If $x - c$ is a factor of $f(x)$, then $f(c) = 0$.

Number of Real Zeros

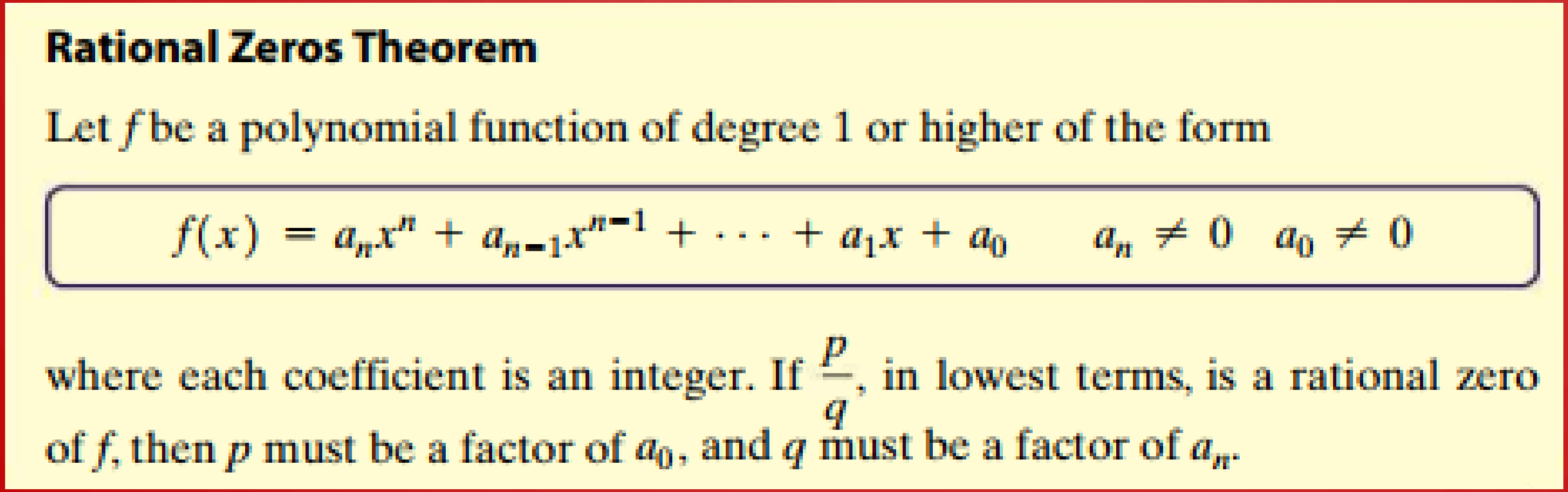
A polynomial function cannot have more real zeros than its degree.

Descartes' Rule of Signs

Let f denote a polynomial function written in standard form.

The number of positive real zeros of f either equals the number of variations in the sign of the nonzero coefficients of $f(x)$ or else equals that number less an even integer.

The number of negative real zeros of f either equals the number of variations in the sign of the nonzero coefficients of $f(-x)$ or else equals that number less an even integer.



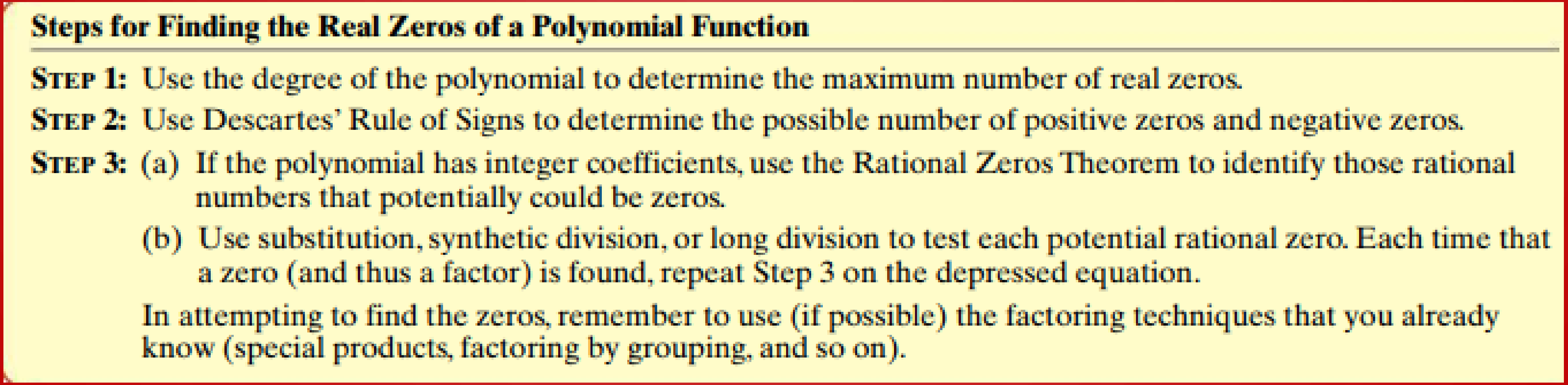
Rational Zeros Theorem

Let f be a polynomial function of degree 1 or higher of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \qquad a_n \neq 0 \quad a_0 \neq 0$$

where each coefficient is an integer. If $\frac{p}{q}$, in lowest terms, is a rational zero of f, then p must be a factor of a_0, and q must be a factor of a_n.

$$Possible\ Rational\ Zeros\ (PRZ) = \frac{p\ which\ is\ all\ factors\ of\ a_0}{q\ which\ is\ all\ factors\ of\ a_n}$$



Steps for Finding the Real Zeros of a Polynomial Function

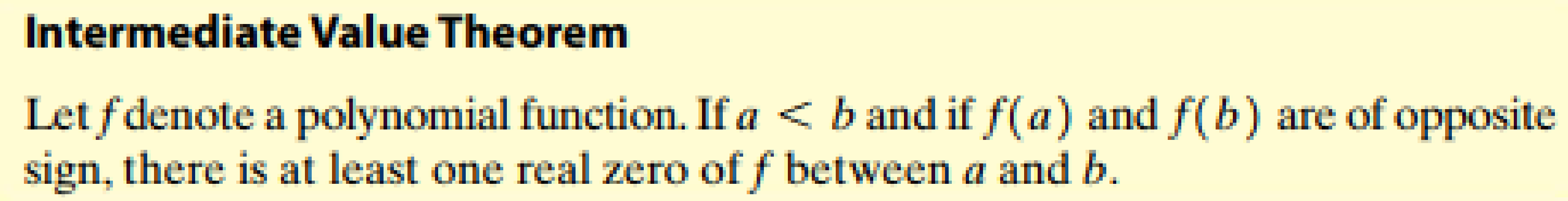
STEP 1: Use the degree of the polynomial to determine the maximum number of real zeros.

STEP 2: Use Descartes' Rule of Signs to determine the possible number of positive zeros and negative zeros.

STEP 3: (a) If the polynomial has integer coefficients, use the Rational Zeros Theorem to identify those rational numbers that potentially could be zeros.

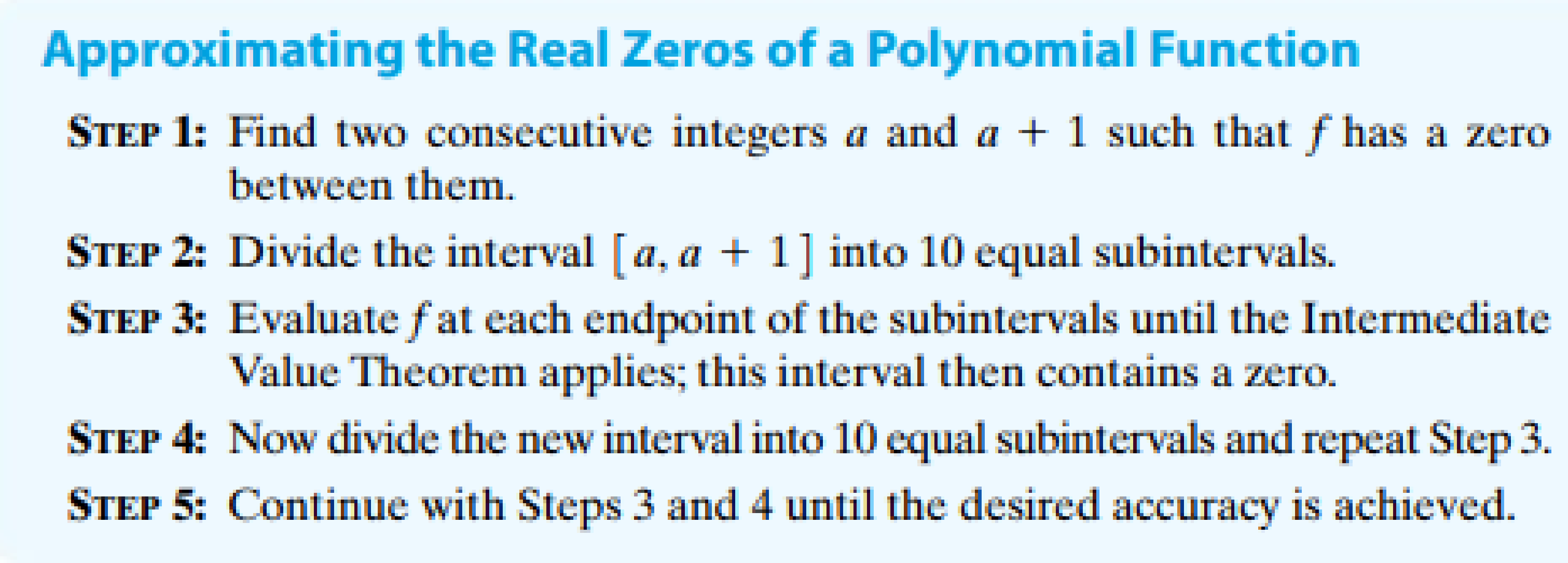
(b) Use substitution, synthetic division, or long division to test each potential rational zero. Each time that a zero (and thus a factor) is found, repeat Step 3 on the depressed equation.

In attempting to find the zeros, remember to use (if possible) the factoring techniques that you already know (special products, factoring by grouping, and so on).



Intermediate Value Theorem

Let f denote a polynomial function. If $a < b$ and if $f(a)$ and $f(b)$ are of opposite sign, there is at least one real zero of f between a and b.



Approximating the Real Zeros of a Polynomial Function

Step 1: Find two consecutive integers a and $a + 1$ such that f has a zero between them.

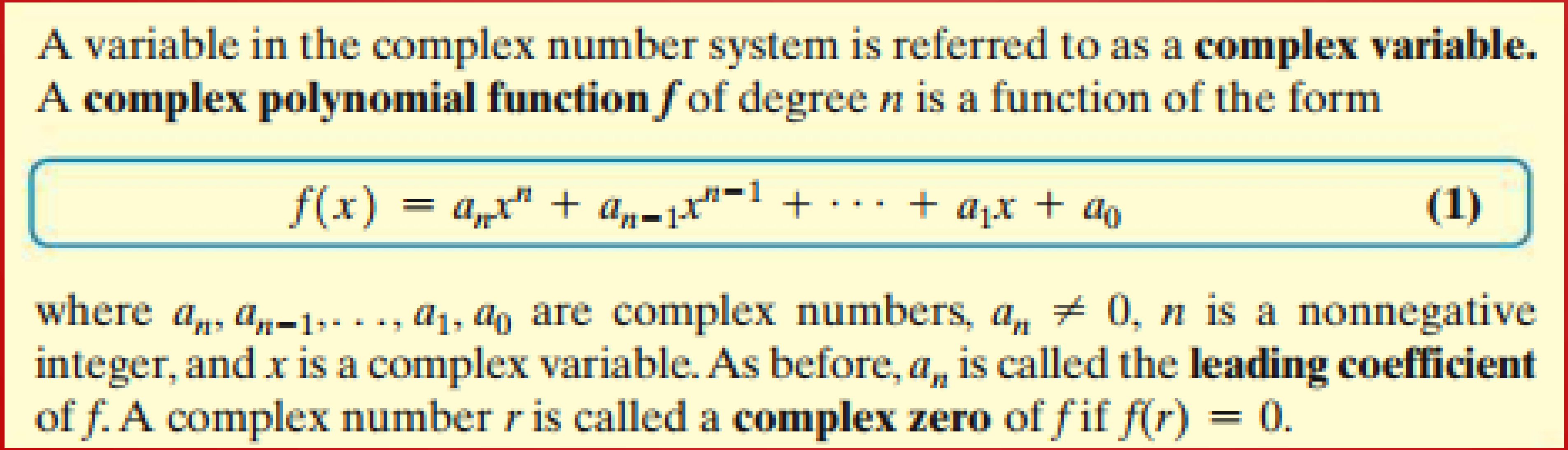
Step 2: Divide the interval $[a, a + 1]$ into 10 equal subintervals.

Step 3: Evaluate f at each endpoint of the subintervals until the Intermediate Value Theorem applies; this interval then contains a zero.

Step 4: Now divide the new interval into 10 equal subintervals and repeat Step 3.

Step 5: Continue with Steps 3 and 4 until the desired accuracy is achieved.

Complex Polynomial

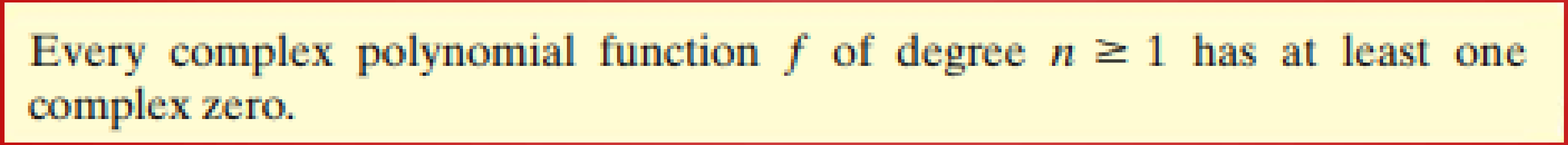


A variable in the complex number system is referred to as a **complex variable.** A **complex polynomial function** f of degree n is a function of the form

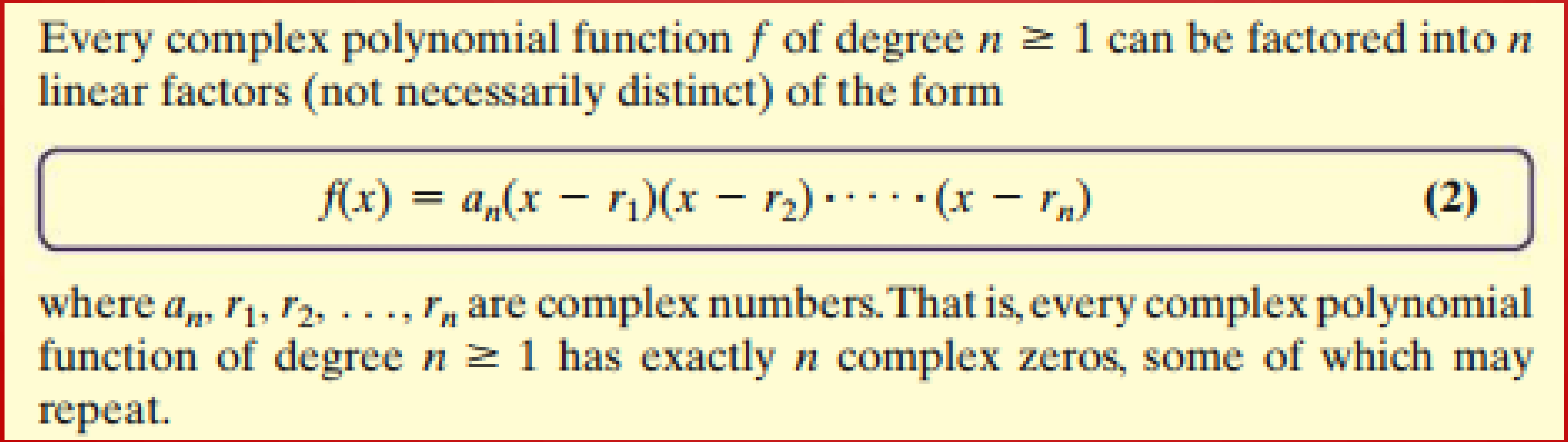
$$\boxed{f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \qquad (1)}$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are complex numbers, $a_n \neq 0$, n is a nonnegative integer, and x is a complex variable. As before, a_n is called the **leading coefficient** of f. A complex number r is called a **complex zero** of f if $f(r) = 0$.

Fundamental Theorem of Algebra (FTA)



Every complex polynomial function f of degree $n \geq 1$ has at least one complex zero.

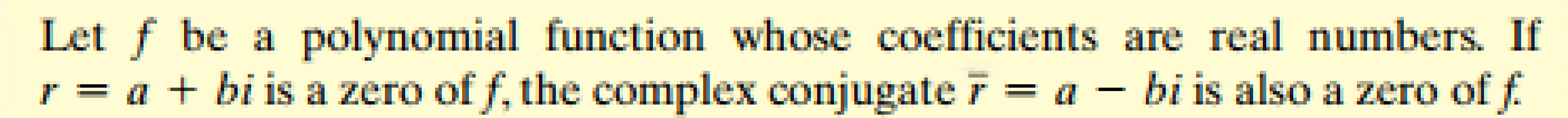


Every complex polynomial function f of degree $n \geq 1$ can be factored into n linear factors (not necessarily distinct) of the form

$$\boxed{f(x) = a_n(x - r_1)(x - r_2) \cdot \cdots \cdot (x - r_n) \qquad \textbf{(2)}}$$

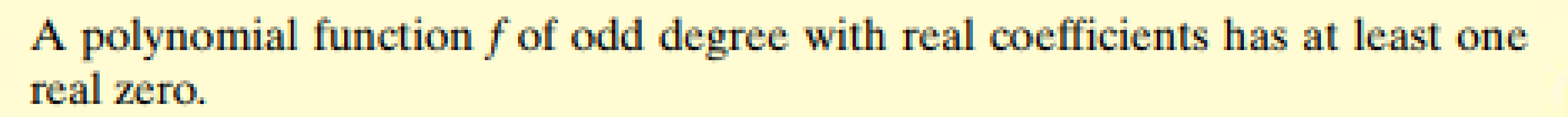
where $a_n, r_1, r_2, \ldots, r_n$ are complex numbers. That is, every complex polynomial function of degree $n \geq 1$ has exactly n complex zeros, some of which may repeat.

Conjugate Pairs Theorem

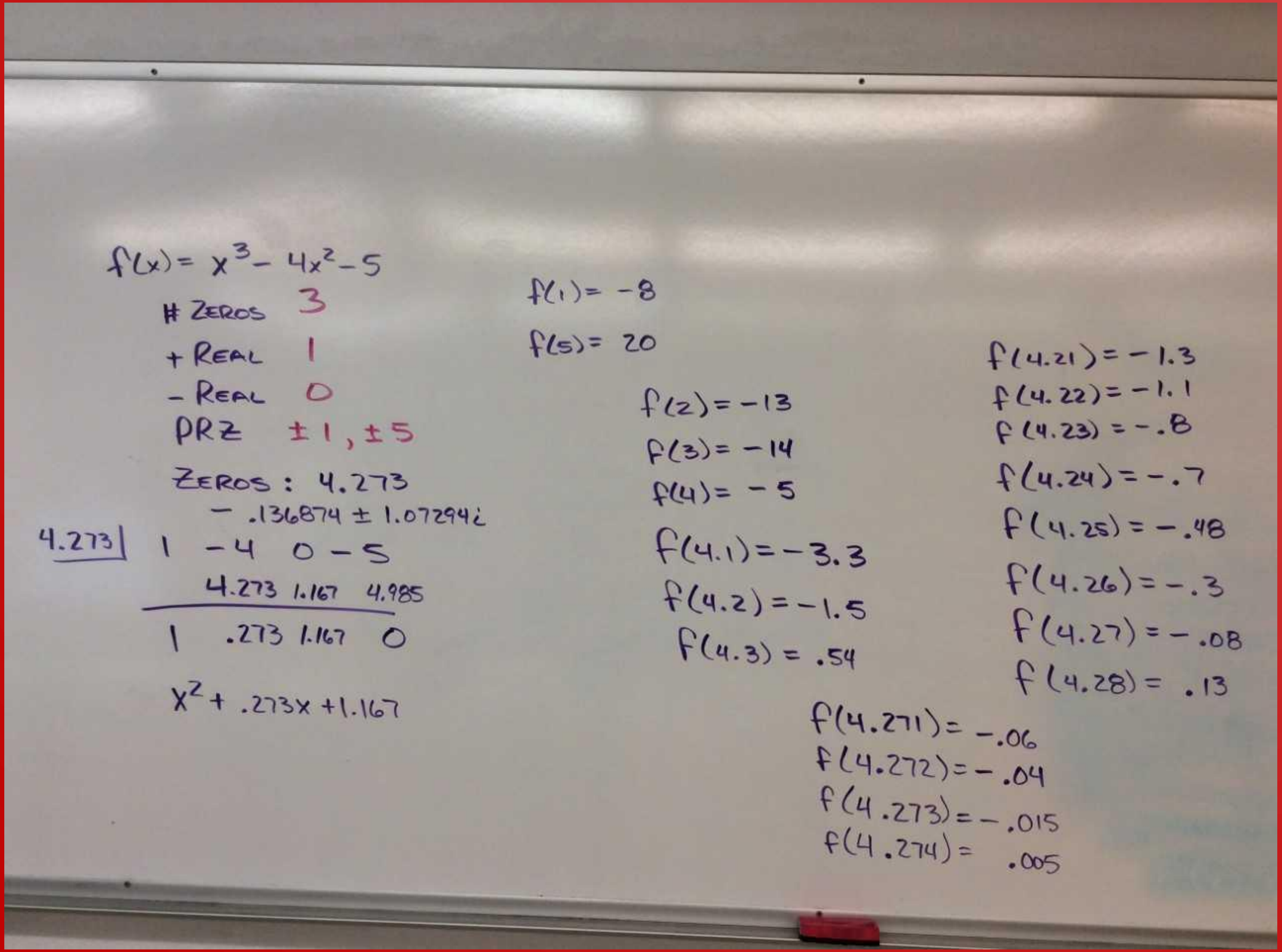


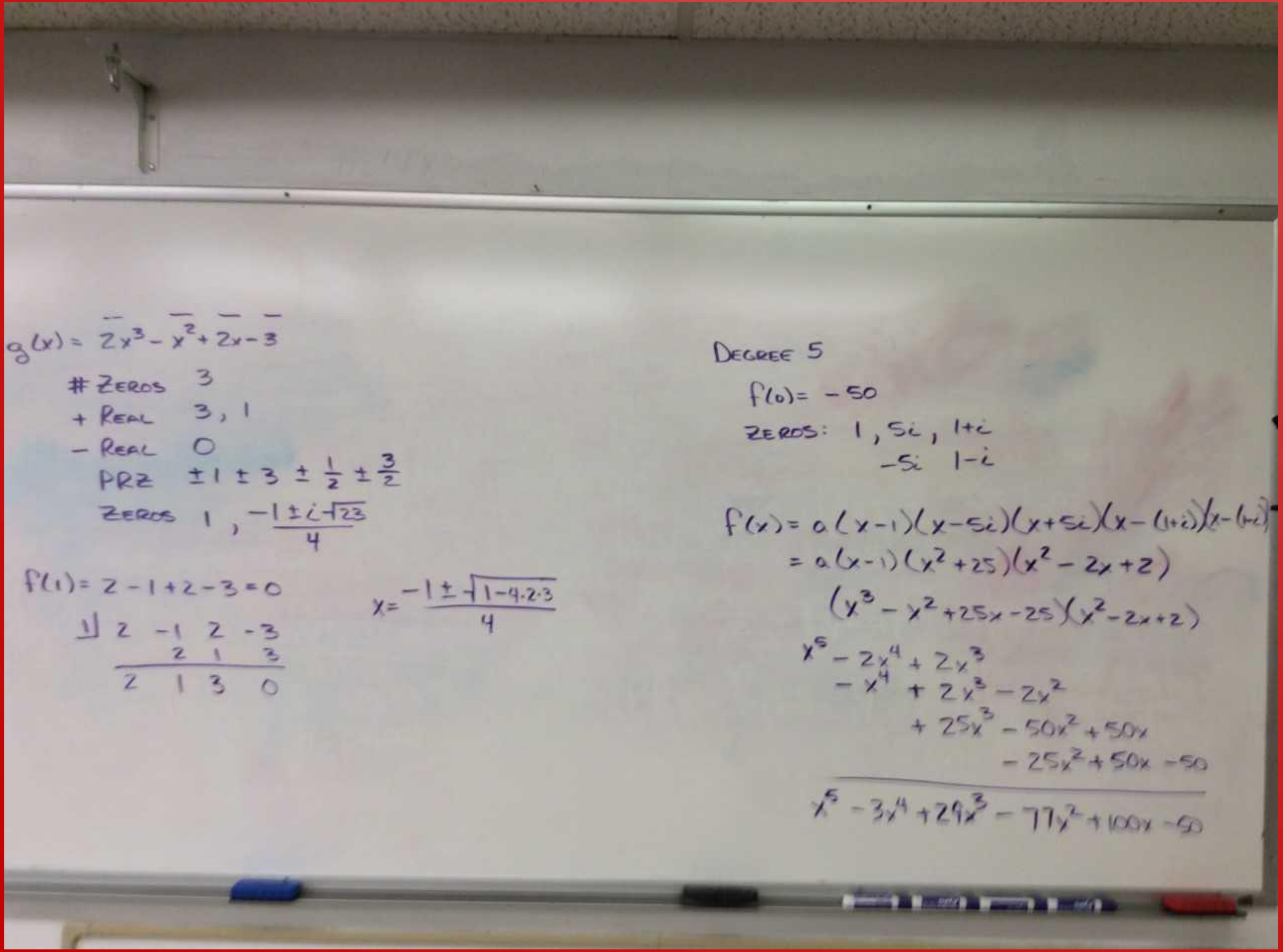
Let f be a polynomial function whose coefficients are real numbers. If $r = a + bi$ is a zero of f, the complex conjugate $\bar{r} = a - bi$ is also a zero of f.

Corollary



A polynomial function f of odd degree with real coefficients has at least one real zero.





$g(x) = \bar{2}x^3 - \bar{x}^2 + \bar{2}x - \bar{3}$

\# ZEROS 3

\+ REAL 3, 1

\- REAL 0

PRZ $\pm 1 \pm 3 \pm \frac{1}{2} \pm \frac{3}{2}$

ZEROS $1, \frac{-1 \pm i\sqrt{23}}{4}$

$f(1) = 2 - 1 + 2 - 3 = 0$

$$\begin{array}{r|rrrr} 1 & 2 & -1 & 2 & -3 \\ & & 2 & 1 & 3 \\ \hline & 2 & 1 & 3 & 0 \end{array}$$

$x = \frac{-1 \pm \sqrt{1 - 4 \cdot 2 \cdot 3}}{4}$

DEGREE 5

$f(0) = -50$

ZEROS: $1, 5i, 1+i$
$-5i \quad 1-i$

$$f(x) = a(x-1)(x-5i)(x+5i)(x-(1+i))(x-(1-i))$$
$$= a(x-1)(x^2+25)(x^2-2x+2)$$
$$(x^3 - x^2 + 25x - 25)(x^2 - 2x + 2)$$

$$\begin{array}{l} x^5 - 2x^4 + 2x^3 \\ \quad - x^4 + 2x^3 - 2x^2 \\ \qquad + 25x^3 - 50x^2 + 50x \\ \qquad\qquad - 25x^2 + 50x - 50 \\ \hline x^5 - 3x^4 + 29x^3 - 77x^2 + 100x - 50 \end{array}$$